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1	2	3	Σ
1TP	3TP	8PP	4TP + 8PP

9. Übung Numerik II

WS 11/12

Mathias Bosewitz
Frederik Garbe
Stephan Gruemmer
Tutor: Maren-Wanda Wolf
Tutorium: Do: 12-14

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3. Aufgabe **Punkte**

(a) Implement the Störmer-Verlet method for this equation in matlab as function

`[p, q, t] = StoermerVerlet(m, g, r0, p0, q0, I, tau)`, where (m, g, r0)

, where $(p0, q0)$, I, and tau denote the problem parameters, the initial values, the time interval and the step size, respectively.

Code:

```
%Störmer-Verlet method für Aufgabe 9.3
function [p,q,t] = StoermerVerlet(m,g,r0,p0,q0,I,tau)

%Anzahl der Iterationen
n=(I(2)-I(1))/tau+1;
t=[I(1)];

%Startwerte
v=[p0/m];
p=[p0];
q=[q0];

%Berechnen der Iterationen des SV-Verfahrens
i=2;
while (i<=n)

    %Speichere t
    t(i)=t(i-1)+tau;

    %Berechne qn+1
    qtemp=q(i-1)+tau*v(i-1)+tau*tau*f93(q(i-1),m,r0,g)/2;

    %Speichere q
    q(i)=qtemp;
```

```
        %Berechne vn+1
        vtemp=v(i-1)+tau/2*(f93(q(i-1),m,r0,g)+f93(q(i),m,r0,g));

        %Speichere v
        v(i)=vtemp;

        %Speichere p
        p(i)=m*vtemp;

        %Iteriere
        i=i+1;
end
```

wobei

```
%Funktion f aus Aufgabe 9.3
function [x] = f93(q,m,r,g)

%m=0.1;
%r=0.1;
%g=1.62;

x=g/r*cos(q);

end
```

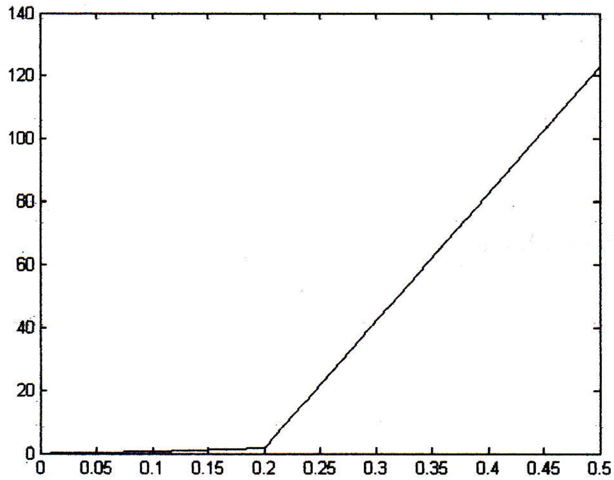
(b) Test your program with the radius $r0 = 10\text{cm}$, the mass $m = 100g$, and the gravity of the moon. Use the time interval $[0s, 20s]$ with the initial value $p0 = 0kgm/s$, $q0 = 0m$ for various time step sizes.

Plots: Für die Schrittweiten $\tau = 0.01, 0.02, 0.05, 0.1, 0.2, 0.5$ ergibt sich:

$t =$ 0.0100 0.0200 0.0500 0.1000 0.2000 0.5000

$e =$ 0.1012 0.1168 0.2267 0.6163 2.0328 122.9408

Maximale Abweichung



Erzeugt durch:

```
%Berechnung der maximalen Abweichung
function [t,e] = testsetting93()

e=[];

[p,q,t]=StoermerVerlet(0.1,1.62,0.1,0,0,[0,20],0.01);
[t,q2]=ode45(@probl93,0:.01:20,[0,0]);
e=[e,err2(q2(:,1),-transpose(q))];

[p,q,t]=StoermerVerlet(0.1,1.62,0.1,0,0,[0,20],0.02);
[t,q2]=ode45(@probl93,0:.02:20,[0,0]);
e=[e,err2(q2(:,1),-transpose(q))];

[p,q,t]=StoermerVerlet(0.1,1.62,0.1,0,0,[0,20],0.05);
[t,q2]=ode45(@probl93,0:.05:20,[0,0]);
e=[e,err2(q2(:,1),-transpose(q))];

[p,q,t]=StoermerVerlet(0.1,1.62,0.1,0,0,[0,20],0.1);
[t,q2]=ode45(@probl93,0:.1:20,[0,0]);
e=[e,err2(q2(:,1),-transpose(q))];

[p,q,t]=StoermerVerlet(0.1,1.62,0.1,0,0,[0,20],0.2);
[t,q2]=ode45(@probl93,0:.2:20,[0,0]);
e=[e,err2(q2(:,1),-transpose(q))];

[p,q,t]=StoermerVerlet(0.1,1.62,0.1,0,0,[0,20],0.5);
[t,q2]=ode45(@probl93,0:.5:20,[0,0]);
e=[e,err2(q2(:,1),-transpose(q))];

t=[0.01,0.02,0.05,0.1,0.2,0.5];
```

```
plot(t,e)

end
```

wobei

```
%Berechnung der maximalen Abweichung
function [e] = err2(x,y)

e=0;
n=size(x,1);
i=1;
while (i<n+1)
    e=max(e,norm(x(i)-y(i)));
    i=i+1;
end

end
```

und

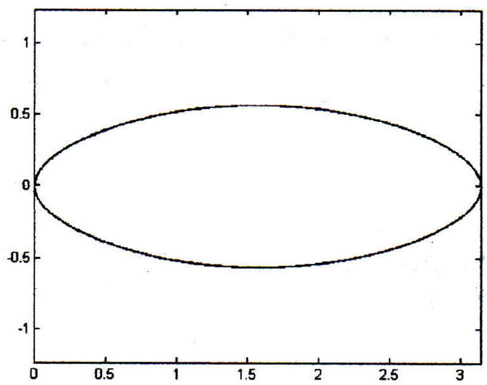
```
function dy = probl93(t , y)
dy=[1/0.1*y(2);-0.1*cos(y(1))*1.62/0.1];

end
```

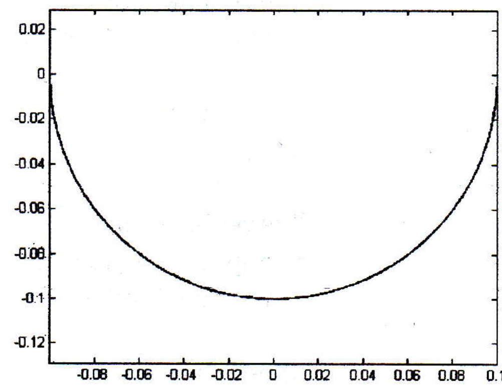
(c) Plot the solution in the phase space and the solution in Euclidean coordinates.

Plot: Der Code

```
[p,q,t]=StoermerVerlet(0.1,1.62,0.1,0,0,[0,20],0.01);
plot(q,p);
axis equal
x=0.1*cos(q);
y=0.1*sin(q);
plot(x,-y);
axis equal
```

liefert:





(d) Solve the Keppler Problem with Störmer-Verlet.

Plot: Unter Verwendung von

$$f(q) = -M^{-1}\nabla U(q) = [-gm_2\frac{(x_{11}-x_{21})}{r_{12}^3} \quad -gm_2\frac{(x_{12}-x_{22})}{r_{12}^3} \quad -gm_2\frac{(x_{13}-x_{23})}{r_{12}^3}$$
$$gm_1\frac{(x_{11}-x_{21})}{r_{12}^3} \quad gm_1\frac{(x_{12}-x_{22})}{r_{12}^3} \quad gm_1\frac{(x_{13}-x_{23})}{r_{12}^3}]$$

Der Code

```
%Störmer-Verlet method für Aufgabe 9.3 Keppler
function [p,q,t] = StoermerVerletKeppler(m1,m2,g,p0,q0,I,tau)

%Anzahl der Iterationen
n=(I(2)-I(1))/tau+1;
t=[I(1)];

%Berechnung der Matrix M
M=[m1,0,0,0,0,0;0,m1,0,0,0,0;0,0,m1,0,0,0;0,0,0,m2,0,0;
0,0,0,0,m2,0;0,0,0,0,0,m2];

%Startwerte
v=[inv(M)*p0];
p=[p0];
q=[q0];

%Berechnen der Iterationen des SV-Verfahrens
i=2;
while (i<=n)

    %Speichere t
    t(i)=t(i-1)+tau;
```

```
    %Berechne qn+1
    qtemp=q(:,i-1)+tau*v(:,i-1)+tau*tau*f93Keppler(q(:,i-1),g,m1,m2)/2;

    %Speichere q
    q(:,i)=qtemp;

    %Berechne vn+1
    vtemp=v(:,i-1)+tau/2*(f93Keppler(q(:,i-1),g,m1,m2)
     +f93Keppler(q(:,i),g,m1,m2));

    %Speichere v
    v(:,i)=vtemp;

    %Speichere p
    p(:,i)=M*vtemp;

    %Iteriere
    i=i+1;
end
```

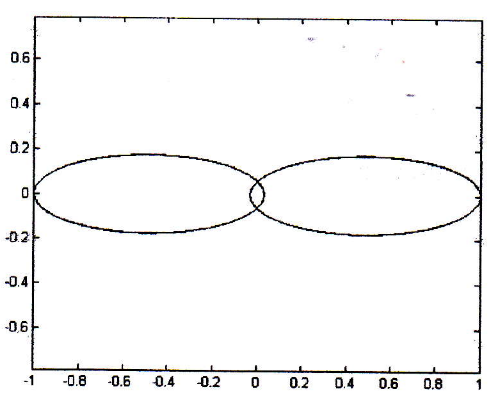
wobei

```
%Funktion f aus Aufgabe 9.3
function [x] = f93Keppler(q,g,m1,m2)

r=g*m1*m2/(((q(1)-q(4))^2+(q(2)-q(5))^2+(q(3)-q(6))^2)^(3/2));

x=[-(q(1)-q(4))*r/m1;-(q(2)-q(5))*r/m1;-(q(3)-q(6))*r/m1;
(q(1)-q(4))*r/m2;(q(2)-q(5))*r/m2;(q(3)-q(6))*r/m2];

end
```

Die Werte

```
[p,q,t] = StoermerVerletKeppler(1000000,1000000,6.67*10^(-11),
[0;1000;0;0;-1000;0],[1;0;0;-1;0;0],[0,1000],0.1);
```

liefern z.B.



5/5 + 3

9.2

$$\Psi\begin{pmatrix}p_n\\q_n\end{pmatrix} := \begin{pmatrix}p_n + \frac{\tau}{2} f(q_n) + \frac{\tau}{2} f\left(q_n + \tau p_n + \frac{\tau^2}{2} f(q_n)\right)\\ q_n + \tau p_n + \frac{\tau^2}{2} f(q_n)\end{pmatrix} = \begin{pmatrix}\tilde{G}\\ \tilde{H}\end{pmatrix}$$

$\tilde{H} \neq$ Hamiltonian sondern nur Schreibweise für Ψ

$$\Rightarrow D\Psi = \begin{pmatrix}\tilde{G}_{p_n} & \tilde{G}_{q_n}\\ \tilde{H}_{p_n} & \tilde{H}_{q_n}\end{pmatrix} =$$

$$= \begin{pmatrix} I + \frac{\tau^2}{2} f'\left(q_n + \tau p_n + \frac{\tau^2}{2} f(q_n)\right) & \frac{\tau}{2} f'(q_n) + \frac{\tau}{2}\left(I + \frac{\tau^2}{2} f'(q_n)\right) f'\left(q_n + \tau p_n + \frac{\tau^2}{2} f(q_n)\right) \\ \tau I & I + \frac{\tau^2}{2} f'(q_n) \end{pmatrix}$$

$\rightarrow J = \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}$

Anm.: $f'(x) \in \mathbb{R}^d$

$$\Rightarrow (D\Psi)^T J (D\Psi) =$$

$$\begin{pmatrix} I + \frac{\tau^2}{2} f'(q_{n+1}) & \tau I \\ \frac{\tau}{2} f'(q_n) + \frac{\tau}{2}\left(I + \frac{\tau^2}{2} f'(q_n)\right) f'(q_{n+1}) & I + \frac{\tau^2}{2} f'(q_n)\end{pmatrix}$$

$$\cdot \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix} \cdot (D\Psi)$$

$$= \begin{pmatrix} -\tau I & \overbrace{I + \frac{\tau^2}{2} f'(q_{n+1})}^{C} \\ -I - \frac{\tau^2}{2} f'(q_n) & \underbrace{\frac{\tau}{2} f'(q_n)}_{:=B} + \underbrace{\frac{\tau}{2}\left(I + \frac{\tau^2}{2} f'(q_n)\right) f'(q_{n+1})}_{:=A}\end{pmatrix} \cdot (D\Psi)$$

$$= \begin{pmatrix} \cancel{-\tau I (I + C)} + \cancel{\tau (I + C)} & -\tau(B + A) + (I + C)(I + \tau B) \\ -(I + \tau B)(I + C) + \tau B + \tau A & \cancel{-(I + \tau B)(B + A)} + \cancel{(B + A)(I + \tau B)}\end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cancel{-\tau B} - \cancel{\tau A} + I + \cancel{C} + \cancel{\tau C B} + \cancel{\tau B} \\ -I - \cancel{\tau B} \underbrace{- C + \tau B C}_{-\tau A} + \cancel{\tau B} + \cancel{\tau A} & \cancel{-B} - \cancel{A} - \cancel{\tau B^2} - \tau B A + \cancel{B} + \cancel{A} + \cancel{\tau B^2} + \tau A B \end{pmatrix}$$

$A = \frac{1}{\tau}(I + \tau B) C$

$\tau BA - \tau AB$
$= BC + \tau B^2 C$
$- CB - \tau BCB$
$= 0$ für $CB = BC$

$$= \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix} = J$$

$\Rightarrow$ symplektisch wenn $f'(q_n)$ und $f'(q_{n+1})$ kommutieren (z.B.: $d = 1$)

$\square$

Ordnung 2:

System: $q' = p$ $\quad x = \begin{pmatrix} p \\ q \end{pmatrix}$

$p' = f(q)$ $\quad x_0 = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$

$f'|_{x=x_0}$

$f(x) = f(x_0) + A(x-x_0) + \underset{\mathcal{O}(\cdot)}{g(x-x_0)}$

$\Rightarrow \quad \|\Psi^{\tau}(x_0) - \Phi^{\tau}(x_0)\| =$ Taylor von Ψ und Φ

$$\left\| \cancel{\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}} + \tau \cancel{\begin{pmatrix} 2f(q_0) \\ -p_0 \end{pmatrix}} + \frac{\tau^2}{2}\begin{pmatrix} A p_0 \\ f(q_0) \end{pmatrix} + \underbrace{\frac{\tau}{2} g\left(\frac{\tau^2}{2} f(q_0)\right)}_{\mathcal{O}(\tau^2)} \right.$$

?

$$- \cancel{\ldots} - \tau \cancel{\begin{pmatrix} f(q_0) \\ p_0 \end{pmatrix}} = \underbrace{e^{\tau\begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}} + \cancel{\begin{pmatrix} A q_0 \\ p_0 \end{pmatrix}}\tau$$

$$= \cancel{\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}} - \tau \cancel{\begin{pmatrix} A q_0 \\ p_0 \end{pmatrix}} - \frac{\tau^2}{2}\begin{pmatrix} A p_0 \\ A q_0 \end{pmatrix}$$

$+ \frac{\tau^2}{2} \quad = + \mathcal{O}(\tau^2) \qquad \left.\right\| = \mathcal{O}(\tau^2)$

$\square$

$\ulcorner \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}^2 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \lrcorner$

Das ist nicht wirklich verständlich!

3/4

9.1 a)

$$y_{n+1} = \Psi^\tau y_n = \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = y_n + \tau \begin{pmatrix} -H_q(p_{n+1}, q_n) \\ H_p(p_{n+1}, q_n) \end{pmatrix}$$

$$\Rightarrow (D\Psi)^T J (D\Psi)$$

$\ulcorner H(p,q) = \frac{1}{2}\langle M^{-1}p, p\rangle + U(q) \lrcorner$

$$= \begin{pmatrix} I + \tau \cdot 0 & -U'' \\ \sim & I + \tau \cdot \underbrace{\frac{d}{dq_n} H_p(p_{n+1}, q_n)}_{=0 \text{ da } H_p \text{ nicht von } q \text{ abh.}} \end{pmatrix}^T J (D\Psi)$$

Nein, denn H_p hängt von p_{n+1} und das von q.

$$= \begin{pmatrix} I & -U'' \\ S & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & S \\ -U'' & I \end{pmatrix}$$

$$= \begin{pmatrix} U'' & I \\ -I & S \end{pmatrix} \begin{pmatrix} I & S \\ -U'' & I \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{U'' - U''} & U''S + I \\ -I + S & \cancel{-S + S} \end{pmatrix}$$

$$= J$$

$$\frac{d}{dp_n} H_p(p_{n+1}, q_n)$$

$$= \frac{d}{dp_n} \langle M^{-1} p_{n+1}, p_{n+1} \rangle$$

$$= \langle M^{-1} \frac{dp_{n+1}}{dp_n}, p_{n+1} \rangle + \langle M^{-1} p_{n+1}, \frac{dp_{n+1}}{dp_n} \rangle$$

$$=: S$$

$S = 0$?

b), c) ?

1/5